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ON SOLUTIONS OF SOME TYPES OF QUATERNIONIC QUADRATIC EQUATIONS

Summary

It is investigated the set of all solutions of any quaternionic quadratic equation of the form $x^2 + xa + bx + c = 0$ by reducing it to one of the form $x^2 + px + q = 0$. This set can be one point, or two points, or a two-dimensional sphere perpendicular to the real axis (but with any centre, unlike the case $x^2 + ax + b = 0$ investigated by A. Pogorui and M. Shapiro in 2004). It is established a necessary and sufficient condition for existence of a spherical solution of a quaternionic quadratic equation of the form $x^2 + ax + b = 0$ and then of the form $x^2 + xa + bx + c = 0$.

1. Introduction

Since the system of quaternions is not commutative, the general form of a quaternionic polynomial equation is much more complicated than one of a polynomial equation in a commutative ring. The former is

$$\sum_{p=1}^{n}\left(\sum_{\ell=1}^{m_p} a_{p,\ell,1}xa_{p,\ell,2}x\ldots a_{p,\ell,p}xa_{p,\ell,p+1}\right) + c = 0 \tag{1}$$

(here and below x is the unknown). Even a quaternionic linear equation can contain a very great number of terms.

The mostly investigated cases of quaternionic polynomial equations are the following ones:

$$\sum_{\ell=1}^{n} a_\ell x^\ell = 0 \tag{2}$$

and (in fact the same)

$$\sum_{\ell=1}^{n} x^{\ell} a_{\ell} = 0. \tag{3}$$

The work [2] contains a theory about solutions of (2) and (3). The authors have proved that the set of all solutions of such equation consists of **isolated points and/or spheres** (so-called spherical solutions).

But a problem about solutions of an **arbitrary** quaternionic polynomial equation is still open. Formally speaking, there exists a universal method to solve any equation of the form (1): one can decompose all known and unknown quaternions by the standard basis of the system of the quaternions and pass to a **system of 4 real** polynomial equations with 4 unknowns. Quaternionic **linear** polynomial equations were investigated by this method in [4]. But as for equations of degree 2 and larger, usage of this method leads to very complicated considerations even if one solves a certain equation and all the more if one constructs a general theory.

So, since all quaternionic linear equations are investigated in [4], the next step should be to investigate **quadratic** ones whose general form is

$$\sum_{\ell=1}^{m} a_{\ell,1} x a_{\ell,2} x a_{\ell,3} + \sum_{p=1}^{n} b_{p,1} x b_{p,2} + c = 0.$$

A particular case

$$ax^2 + bx + c = 0 \tag{4}$$

(and also $x^2a+xb+c=0$) is already investigated in [2] because it is a particular case of (2) (respectively, (3)). Nevertheless [2] gives very general theory, and it is possible to clarify some details. Some simple investigations for several types of equations of the form (4) were done in [3] (e. g., a direct formula for the solutions when a and b are real). Our paper makes one more step in investigations of the case: we clarify when such equation has a spherical solution.

Another field for the nearest further investigations is to describe sets of solutions of other types of quaternionic quadratic equations (different from $ax^2 + bx + c$ and $x^2a + xb + c$). This paper makes only a little step in this direction covering a small part of the set of all quaternionic quadratic equations. But we believe that it is a good beginning for future investigations and that our results are interesting by itself.

Let us pass to brief description of the content of each section of the paper.

In Section 2 we clarify terminology and notations of the paper in order to avoid ambiguousness and also to explain the main notions and notations for those readers who are not enough familiar with the area.

In Section 3 we investigate equations of the form

$$x^2 + xa + bx + c = 0. \tag{5}$$

It turns out that (5) can be reduced to an equation of the kind (4) by a very simple idea: to shift the unknown. Equation (5) represents a narrow class of quaternionic quadratic equations, but we establish a property showing that from some point of view this class is "much more general" than (4). Namely, now the set of solutions is of the same type as in [2], but **without restrictions on locations** of the points and the spheres: while for (4) the centres of the spherical solutions are always **on the real axis**, we prove that **every quaternion** is such centre for some equation of the form (5); as for the isolated points, somewhat similar observation also arises. In Section 4 we prove a necessary and sufficient conditions for an equation of the form (4) (and then (5) as a corollary) to have a spherical solution. In the proof we use an interesting idea which in fact allows to restrict a quadratic equation to a linear one for the case of a spherical solution. It turns out that the spherical solutions arise comparatively seldom.

2. Terminology and notations of the paper

The letter $\mathbb{H}$ denotes the system of (real) quaternions, and, of course, the letters $\mathbb{R}$ and $\mathbb{C}$ are used in their usual sense (the systems of real and complex numbers respectively).
We use standard notations i, j, k for the quaternionic imaginary units; recall that

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Moreover if

$$x = x_0 + x_1 i + x_2 j + x_3 k, \quad x_0, x_1, x_2, x_3 \in \mathbb{R}$$

then the number x_0 is called the scalar part of x and is denoted by

$$\operatorname{Sc} x; \quad \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$$

is called the modulus of x and is denoted by $|x|$; $x_0 - x_1 i - x_2 j - x_3 k$ is called the *conjugate* number with respect to x and is denoted by $\overline{x}$.
We will often speak about spheres in $\mathbb{H}$. We will use the word "*sphere*" for a usual sphere only, i. e., being a two-dimensional surface and the boundary of a three-dimensional ball (of course, it is necessary to use the usual representation of quaternions as points of the 4-dimensional geometric space). Moreover we will often speak about a sphere *with a constant scalar component*; this means that every quaternion from the sphere has the same scalar part; by other words, the sphere belongs to a hyperplane perpendicular to the real axis.

3. The sets of the solutions of quaternionic quadratic equations of a certain type

Let us consider in $\mathbb{H}$ any equation of the following form:

$$x^2 + xa + bx + c = 0, \tag{6}$$

where x is the unknown (from $\mathbb{H}$), a, b, c are given quaternions. It is easy to show by simple direct calculations that this equation can be rewritten as follows:

$$(x+a)^2 + (b-a)(x+a) + (c-ba) = 0. \tag{7}$$

Note that (7) can be obtained from (6) by the following two operations: firstly one separates a square

$$(x+a)^2 = x^2 + xa + ax + a^2$$

and gets

$$(x+a)^2 + (b-a)x + (c-a^2) = 0;$$

then one rewrites the linear part of the last equation as a linear function of $x+a$.

Putting $x+a=y$ we rewrite (7) as follows:

$$y^2 + (b-a)y + (c-ba) = 0. \tag{8}$$

We have obtained a particular case of the equation (2) which has being investigated in [2]. According to Theorem 6 from [2], the set of all solutions of (2) consists of isolated points and/or spheres with constant scalar components and with the centres on the real axis; moreover the number of the isolated points together with the doubled number of the spheres is not larger than the degree of the equation. Now the degree equals 2 and thus the set is or one point, or two points, or one sphere. This set cannot be empty due to Theorem 4 from [2], which claims that any equation of the form (2) has at least one solution (if its degree does not equal 0).

So, we easily obtain the following result about any equation of the form (6):

Theorem 1. *Let an equation of the form (6) be given, where $a, b, c \in \mathbb{H}$. Then the set of all its quaternionic solutions is or one point, or two points, or one sphere with a constant scalar component (a so-called spherical solution).*

Though (6) represents a so narrow class of quaternionic quadratic equations, it turns out that in some sense this class is "very general". Namely, now we will formulate a theorem about existence of an equation of the form (6) with a given solution. After the proof of this theorem we will explain why (6) can be called "much more general" than (4). But note just now that due to shifting of the unknown Theorem 1 does not demand for the sphere to be with the centre on the real axis.

Theorem 2. 1. *For every pair of quaternions s and t there exist quaternions a, b, c such that (6) has exactly two solutions which are equal to s and t.*

2. *For every quaternion s there exist quaternions a, b, c such that (6) has exactly one solution which is equal to s.*

3. *For every pair $s \in \mathbb{H}$ and $r > 0$ there exist quaternions a, b, c such that the set of solutions of (6) is the sphere with a constant scalar component, with the centre s and the radius r.*

Proof. As for the first proposition, it is obvious that the equation

$$(x-s)(x-t)=0$$

is appropriate (due to absence of zero divisors in $\mathbb{H}$); it can be rewritten as

$$x^2 - xt - sx + st = 0.$$

Analogously an equation for the second proposition is

$$(x-s)^2 = 0 \quad \text{or} \quad x^2 - xs - sx + s^2 = 0.$$

In order to prove the third proposition take the following equation:

$$x^2 + r^2 = 0.$$

It is known (see, e. g., [1]) that the set of its solutions is a sphere with a constant scalar component, with the centre 0 and the radius r. Then, obviously, the equation

$$(x-s)^2 + r^2 = 0$$

is appropriate; it can be rewritten as

$$x^2 - xs - sx + (s^2 + r^2) = 0.$$

□

Recall that, according to [2], the centres of spherical solutions of a quaternionic polynomial equation **with only left coefficients** are always **on the real axis**. So, the form (6) is "much more general" from this "inverse problem" point of view: the centre can be anywhere in $\mathbb{H}$. Moreover note that the first proposition of the theorem refers to every pair of quaternions. But in the case with only left coefficients it cannot be a pair from one and the same sphere with a constant scalar component and with the centre on the real axis because then, according to Proposition 2 from [2], there would be not only two but infinitely many solutions (all this sphere).

Example 1. Let us solve the following equation in $\mathbb{H}$:

$$x^2 + xj + kx = 0.$$

According to the procedure described above, we pass to the following equation:

$$y^2 + (k-j)y + i = 0, \tag{9}$$

where $y = x + j$.

According to the procedure of [2], we construct the so-called basic polynomial F^* for (9). We will try to describe the procedure by a way convenient for the reader to understand rules from [2] (if he or she is not familiar with this work), but we will avoid strict formulations of the rules. So, firstly it is necessary to do the following calculation:

$$(z^2 + z\overline{(k-j)} + \bar{i})(z^2 + (k-j)z + i),$$

considering z as a real number (though at the next step it is necessary to treat z as a complex number). So, the calculation gives:

$$F^*(z) = z^4 + 2z^2 + 1 = (z^2+1)^2.$$

Obviously, this polynomial has only one pair of mutually conjugate complex zeros: i and $-i$ (they are of multiplicity 2). According to [2], one has to look for solutions of (9) on the sphere with a constant scalar component, with the centre 0 and the radius 1 (the centre is defined by the real component of a zero of F^*, the radius is defined by its imaginary one). It is easy to check that j is a solution of (9). Take another number from this sphere, e. g., i. It is easy to see that i is not a solution of (9). According to Proposition 2 from [2], we have got enough information to claim that (9) has only one solution: j.

Then we easily pass from y to x and obtain the solution of the initially given equation:

$$x = 0.$$

Of course, this solution is obvious, but we have found out that there is no other solution.

Remark 1. Example 1 proves that the authors of [2] mistook writing in Corollary 5 about **one-to-one** correspondence between zeros of multiplicity 2 of the basic polynomial and spherical solutions of a given equation: now we have a zero of multiplicity 2 but no spherical solution. In fact the authors of [2] meant that the number of spherical solutions of the given equation is **not larger** than the number of multiple zeros of the corresponding basic polynomial.

Example 2. Let us solve the following equation in $\mathbb{H}$:

$$x^2 + x(1+j) + jx + (1+j) = 0.$$

According to the procedure, one obtains the following:

$$y^2 - y + 2 = 0, \tag{10}$$

where $y = x + 1 + j$. All coefficients of (10) are real. Therefore the corresponding basic polynomial can be written as a square:

$$F^*(z) = (z^2 - z + 2)^2.$$

Again we have got a polynomial with zeros of multiplicity 2. They are

$$\frac{1}{2} + \frac{\sqrt{7}}{2}i \quad \text{and} \quad \frac{1}{2} - \frac{\sqrt{7}}{2}i.$$

Obviously, these two numbers are also solutions of (10) (if one treats them as quaternions). Then due to Proposition 2 from [2] every quaternion from the sphere with a constant scalar component, with the centre $\frac{1}{2}$ and the radius $\frac{\sqrt{7}}{2}$ is a solution of (10). It is clear that this sphere is all the set of the solutions. So, for the initial equation we have the following set of solutions: the sphere with a constant scalar component, with the centre $-\frac{1}{2} - j$ and the radius $\frac{\sqrt{7}}{2}$. By other words,

$$x = -\frac{1}{2} + \alpha i + (\beta - 1)j + \gamma k,$$

where

$$\alpha, \beta, \gamma \in \mathbb{R} \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2 = \frac{7}{4}.$$

Remark 2. It is clear that the method to reduce an equation of the form (6) to one of the form (4) can be applied to equations in any non-commutative hypercomplex system, not only in $\mathbb{H}$. Therefore any theory about equations of the form (4) in any such system allows to construct at once the corresponding theory about equations of the form (6).

4. A necessary and sufficient condition for existence of a spherical solution

In this section we investigate a question about when a quaternionic quadratic equation has a spherical solution.

We will use the following obvious relation:

$$\forall x \in \mathbb{H} \qquad x^2 = 2(\operatorname{Sc} x)x - |x|^2. \tag{11}$$

Theorem 3. *Let the following quaternionic equation be given:*

$$x^2 + ax + b = 0, \tag{12}$$

where x is the unknown, a, b are given quaternions. Then (12) *has a spherical solution if and only if the following two conditions are satisfied:*

1) *both the coefficients a and b are real numbers;*

2) $a^2 - 4b < 0$.

Proof. Let us suppose firstly that (12) has a spherical solution. According to [2] (taking into attention Remark 1), the basic polynomial has a multiple zero.

It is known from [2] that the basic polynomial has only real coefficients and its degree equals 4 (twice as much as the degree of the given equation). Note first of all the following: if all zeros of the basic polynomial are real then, according to [2], (12) cannot have any spherical solution because the radius of any such sphere would equal 0.

So, the basic polynomial has at least one non-real complex zero. Denote this zero by α. Then $\overline{\alpha}$ is also a zero of the basic polynomial. Suppose firstly that the basic polynomial has also a real zero ρ. Then just this ρ has to be the multiple zero of the basic polynomial (we take into attention that its degree equals 4). Then we have listed all its complex zeros: α, $\overline{\alpha}$, and ρ. As we already said, a real zero of the basic polynomial cannot correspond to a spherical solution of (12) (radius 0). But the pair of the zeros α and $\overline{\alpha}$ also cannot correspond to the spherical solution because these zeros of the basic polynomial are not multiple. Then (12) has no spherical solution. This contradiction proves that every zero of the basic polynomial is non-real.

So, the only possible situation now is the following: the basic polynomial has only two complex zeros, both being of multiplicity 2, non-real and mutually conjugate; denote these zeros by ζ and $\overline{\zeta}$. Write down the basic polynomial:

$$F^*(z) = z^4 + (\overline{a} + a)z^3 + (\overline{b} + \overline{a}a + b)z^2 + (\overline{a}b + \overline{b}a)z + \overline{b}b,$$

or

$$F^*(z) = z^4 + 2(\operatorname{Sc} a)z^3 + (|a|^2 + 2\operatorname{Sc} b)z^2 + (\overline{a}b + \overline{b}a)z + |b|^2.$$

Using the Viète formulas we can write:

$$\begin{cases} 2(\zeta + \overline{\zeta}) = -2\operatorname{Sc} a, \\ (\zeta\overline{\zeta})^2 = |b|^2, \end{cases}$$

or

$$\begin{cases} \operatorname{Re}\zeta = -\dfrac{\operatorname{Sc} a}{2}, \\ |\zeta|^2 = |b|. \end{cases}$$

Let us pass to solutions of (12). According to [2],

$$\operatorname{Sc} x = \operatorname{Re}\zeta \quad \text{and} \quad |x| = |\zeta|$$

(for every x satisfying (12)). Therefore

$$\begin{cases} \operatorname{Sc} x = -\dfrac{\operatorname{Sc} a}{2}, \\ |x|^2 = |b|. \end{cases} \tag{13}$$

Then, substituting (11) into (12) we write:

$$2(\operatorname{Sc} x)x - |x|^2 + ax + b = 0; \tag{14}$$

and substituting (13) into (14) we write:

$$(a - \operatorname{Sc} a)x + (b - |b|) = 0.$$

We have obtained a usual (with only left coefficients) quaternionic **linear** equation. It is well-known that such equation has only one solution if $a - \operatorname{Sc} a \neq 0$. But, according to our assumption, there exist infinitely many such x (a whole sphere). Thus $a - \operatorname{Sc} a = 0$, so that $a \in \mathbb{R}$. Moreover $b - |b| = 0$, otherwise the equation would not have any solution. Thus $b \in \mathbb{R}$, and the first condition is proved.

So, now (12) is an equation with only real coefficients. Therefore the corresponding basic polynomial is

$$F^*(z) = (z^2 + az + b)^2.$$

Obviously, its zeros are the same as complex solutions of (12) (but with doubled multiplicity). Taking again into attention that existence of a non-real solution is necessary, we conclude that the discriminant of (12) is negative, but it is the same as the second condition from the theorem. So, the proof of the necessary condition for existence of a spherical solution of (12) is complete.

Then let us suppose that $a, b \in \mathbb{R}$ and $a^2 - 4b < 0$. We have to prove that (12) has a spherical solution. But considering (12) in $\mathbb{C}$ we obtain an equation with mutually conjugate non-real solutions. These solutions are also solutions of (12) considered in

$\mathbb{H}$, and they both belong to one and the same sphere with a real centre and with a constant scalar component. According to Proposition 2 from [2], every point of this sphere is a solution of (12), and the theorem is proved. □

Corollary 1. *Let the following quaternionic equation be given:*

$$x^2 + xa + bx + c = 0, \tag{15}$$

where x is the unknown, a, b, c are given quaternions. Then (15) *has a spherical solution if and only if the following two conditions are satisfied:*

1) *$b - a$ and $c - ba$ are real numbers;*
2) *$(a-b)^2 + 4(ba - c) < 0$ (or, that is the same, $a^2 + b^2 - ab + 3ba - 4c < 0$).*

For the proof it is enough to take into attention the consideration of the beginning of Section 3 and apply Theorem 3 to (8). □

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O ROZWIĄZANIACH PENYCH TYPÓW KWADRATOWYCH RÓWNAŃ KWATERNIONOWYCH

S t r e s z c z e n i e

Wyznaczamy klasę wszystkich rozwiązań dowolnego kwadratowego równania kwaternionowego postaci $x^2+xa+bx+c=0$ przez zredukowanie go do postaci $x^2+px+q=0$. Ta klasa może redukować się do punktu albo dwóch punktów, albo do dwuwymiarowej sfery prostopadłej do osi rzeczywistej (ale z dowolnym środkiem, a nie jak w przypadku $x^2+ax+b=0$ badanym przez A. Pogoruja i M. Shapirę w 2004 r.). Znaleziono warunek konieczny i wystarczający dla istnienia rozwiązania sferycznego kwadratowego równania kwaternionowego postaci $x^2+ax+b=0$ oraz $x^2+xa+bx+c=0$.